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Group Theoretic Method for a Converging Shock Wave Problem

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Abstract—Group theoretic method is used to study the classes of cylindrically symmetric similar motion of a nonequilibrium gasdynamic flow under the influence of magnetic field. The profiles of flow and field variables have been made by performing numerical computations which enable one to assess the influence of magnetic field on the flow pattern.

Keywords—Group theoretic method, Lie derivative, Similarity solution, Nonequilibrium gasdynamics, Magnetic field, Converging shock wave.

1. INTRODUCTION

Recently, a great deal of attention has been focused on the use of group theoretic methods because of their wide applications in determining analytical solutions of nonlinear differential equation(s) of physical and engineering interests. With its origin in the pioneering work of Lie [1], group theoretic methods received much impetus through the work of Bluman and Cole [2], Ovsjannikov [3], Olver [4]. These methods have gained a popularity primarily due to the existence of transformation(s) of variables which achieves a reduction in the number of independent variables in a system of equations.

The basic idea of these methods is to find a Lie group of transformations under which partial differential equations are invariant. The conjection of these transformations reduces the number of variables of partial differential equations and the invariants of the group become the new variables. The general group theoretic methods described in [2,3,5] guarantee the complete determination of the invariance group, and consequently, all self-similar solutions to the problem, as a by-product, we are able to characterize analytically the general form of the rate of change of vibrational

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energy in a nonequilibrium magneto-gas-dynamic flow under which the entire system of partial differential equations admits a local Lie group of transformations, and hence, similarity solution.

The gases attain very high temperature due to the motion in nonequilibrium gas-dynamic flows. Since at such a high temperature, a gas is likely to be ionized, the effect of magnetic field may also be significant in the study of converging shock waves in nonequilibrium gas-dynamic flows.

2. MATHEMATICAL FORMULATION

Under the assumption that the magnetic field is azimuthal, the set of nonlinear partial differential equations governing the nonequilibrium magneto-gas-dynamic flow in a cylinderical symmetry are given by

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \frac{\rho u}{x} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \rho^{-1}\left(\frac{\partial p}{\partial x} + \frac{\partial b}{\partial x}\right) = 0, \tag{2.2}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a_f^2\left(\frac{\partial u}{\partial x} + \frac{u}{x}\right) + \left(-\frac{h_q}{h_\rho}a_f^2\right)W(p,\rho,q) = 0, \tag{2.3}$$

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + 2b\frac{\partial u}{\partial x} + \frac{2bu}{x} = 0, \tag{2.4}$$

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = W(p,\rho,q), \tag{2.5}$$

$$p = \rho RT, \tag{2.6}$$

where x is the spacial coordinate of time, ρ denotes the gas density, p is the pressure, u is the particle velocity in the direction of the piston, $h = h(p,\rho,q)$ is the enthalpy, b is the magnetic pressure defined as $b = \mu B^2/2$ (where μ is magnetic permeability and B is the azimuthal magnetic field), $a_f^2 = \gamma_f(q)(p/\rho) = (h_\rho/(1/\rho - h_p))$, where $\gamma_f(q) = (4+\mathrm{q})/3$. The quantity W, which is a known function of p, ρ, and q denotes the rate of change of vibrational energy and is given by [7]

$$W(p,\rho,q) = \frac{\{q_e(p,q) - q\}}{\tau_v(p,q)}, \tag{2.7}$$

where q is vibrational energy and q_e is the equilibrium value of q defined as $q_e = q_{e_0}(x) + C(p,\rho)R(T - T_0(x))$; here T is the translational temperature, R is the specific gas constant, suffix '0' refers to the initial rest condition, and the quantities τ_v and C are the relaxation time and the ratio of vibrational specific heat to the specific heat constant.

The Rankine-Hugoniot jump conditions, which hold across the shock-front as the shock discontinuity propagates into the medium, act as boundary conditions for the problem. In order to make the jump conditions more tractable for the similarity analysis, we make the strong shock assumption. Let $x = X(t)$ be the shock path and V be the shock velocity, then jump conditions are given by

$$\begin{gathered} u(X(t),t) = \frac{2}{\gamma_f+1}V, \qquad \rho(X(t),t) = \frac{\gamma_f+1}{\gamma_f-1}\rho_0(X(t)), \qquad q(X(t),t) = q_0(X(t)), \\ p(X(t),t) = \frac{2}{\gamma_f+1}\rho_0(X(t))V^2, \quad \text{and} \quad b(X(t),t) = \frac{1}{2}\left(\frac{\gamma_f+1}{\gamma_f-1}\right)^2 c_0\rho_0(X(t))V^2, \end{gathered} \tag{2.8}$$

where $c_0 = [2b_0/\rho_0(X(t))V^2]$ is the shock cowling number and the medium just ahead of the shock is specified by $u_0 = 0$, $p_0 = \text{constant}$, $\rho_0 = \rho_0(x) = \rho_c(x/x_0)^\theta$, $b_0 = \text{constant}$, $q_0 = q_0(x)$.

3. INVARIANCE GROUPS

The set of partial differential equations governing the flow is of quasilinear type. In general, it is very difficult to find solution of this system without approximations. Here, we assume that there exists a solution of the system of partial differential equations subject to jump conditions along a family of curves, called similarity curves. Further, we assume that the shock trajectory is embedded in the family of similarity curves. This type of solution is called a similarity solution.

The idea to find invariance groups is to find one parameter infinitesimal group of transformations

$$\begin{aligned} &\bar{t} = t + \epsilon\psi(x,t,\rho,u,p,b,q), && \bar{x} = x + \epsilon\chi(x,t,\rho,u,p,b,q), \\ &\bar{\rho} = \rho + \epsilon S(x,t,\rho,u,p,b,q), && \bar{u} = u + \epsilon U(x,t,\rho,u,p,b,q), \\ &\bar{p} = p + \epsilon P(x,t,\rho,u,p,b,q), && \bar{b} = b + \epsilon \Pi(x,t,\rho,u,p,b,q), \\ &\bar{q} = q + \epsilon E(x,t,\rho,u,p,b,q), \end{aligned} \tag{3.1}$$

where ψ, χ, S, U, P, Π, and E are the generators and functions of x, t, ρ, u, p, b, and q, under which the system of partial differential equations (2.1) through (2.6) and jump conditions (2.8) are invariant. The entity ϵ is so small that its square and higher powers may be neglected. If it is possible to find such a group, then the standard techniques [2] permit the number of independent variables in the problem to be reduced by one, there by allowing the system of partial differential equations to be replaced by a system of ordinary differential equations.

To carry out the analysis, we introduce the notation $x_1 = t$, $x_2 = x$, $u_1 = \rho$, $u_2 = u$, $u_3 = p$, $u_4 = b$, $u_5 = q$, and $p^i_j = \frac{\partial u_i}{\partial x_j}$, where $i = 1, 2, \dots, 5$ and $j = 1, 2$.

A system of differential equations

$$H_n\left(x_j, u_i, p^i_j\right) = 0, \qquad n = 1, 2, \dots, 5,$$

is said to be constantly conformally invariant under the infinitesimal group (3.1), if there exists constants $\alpha_{ns} (n, s = 1, 2, \dots, 5)$, such that [6]

$$\mathcal{L}H_n = \alpha_{ns} H_s, \qquad n = 1, 2, \dots, 5, \tag{3.2}$$

for all smooth surfaces $u_i = u_i(x_j)$, where $\mathcal{L}$ is the Lie derivative in the direction of the extended vector field

$$\mathcal{L} = \xi^i_x \frac{\partial}{\partial x_j} + \xi^i_u \frac{\partial}{\partial u_i} + \xi^i_{p_j} \frac{\partial}{\partial p^i_j},$$

where $\xi^1_x = \psi$, $\xi^2_x = \chi$, $\xi^1_u = S$, $\xi^2_u = U$, $\xi^3_u = P$, $\xi^4_u = \Pi$, $\xi^5_u = E$, and

$$\xi^i_{p_j} = \frac{\partial \xi^i_u}{\partial x_j} + \frac{\partial \xi^i_u}{\partial u_k} p^k_j - \frac{\partial \xi^l_x}{\partial x_j} p^i_l - \frac{\partial \xi^l_x}{\partial u_n} p^i_l p^n_j, \qquad l = 1, 2, \quad n = 1, 2, 3, 4, 5 \tag{3.3}$$

are the generators of the derivative transformation.

Now equation (3.2) gives

$$\frac{\partial H_n}{\partial x_j} \xi^i_x + \frac{\partial H_n}{\partial u_i} \xi^i_u + \frac{\partial H_n}{\partial p^i_j} \xi^i_{p_j} = \alpha_{ns} H_s, \qquad n = 1, 2, 3, 4, 5. \tag{3.4}$$

Substitution of $\xi^i_{p_j}$ from equation (3.3) into (3.4) gives a polynomial in p^i_j. Setting the coefficients of p^i_j and $p^i_j p^k_l$ to zero gives a system of first-order linear partial differential equations in the generators ψ, χ, S, U, P, Π, and E. This system is called the system of determining equations of the group which can be solved to find the invariance group (3.1). Carrying out this program for the conservation law equations (2.1)–(2.5), we obtain the most general group under which this system is invariant.

We can summarize our main result as follows.

THEOREM 1. *Under the polytropic gas, strong shock, and azimuthal magnetic field assumption, the equation for nonequilibrium magneto-gas-dynamic flow (equations (2.1)–(2.5)) are constantly conformally invariant under the infinitesimal group (3.1) with the generators given by*

$$\begin{gathered} S=(\alpha_{11}+a)\rho, \quad U=(\alpha_{22}+a)u, \quad P=(2\alpha_{22}+\alpha_{11}+3a)p, \quad \Pi=(2\alpha_{22}+\alpha_{11}+3a)b, \\ E=2(\alpha_{22}+a)q+d, \quad \psi=at+d, \quad \text{and} \quad \chi=(\alpha_{22}+2a)x, \end{gathered} \tag{3.5}$$

where a and d are constants of integration.

The rate of change of vibrational energy W is to satisfy the following partial differential equation:

$$S\frac{\partial W}{\partial \rho}+P\frac{\partial W}{\partial p}+E\frac{\partial W}{\partial q}=(2\alpha_{22}+a)W, \tag{3.6}$$

which may be integrated by the method of characteristics to find the general form of $W(\rho,p,q)$. The integration of equation (3.6) leads to

$$W=\left(p^{(2\delta-3)a/\alpha_{11}+(2\delta-1)a}\right)q(\eta,\zeta), \tag{3.7}$$

where q is an arbitrary function of η and ζ, where $\eta=\rho p^{-(\alpha_{11}+a)/(\alpha_{11}+(2\delta-1)a)}$, $\delta=(\alpha_{22}+2a)/a$, and $\zeta=p(q+d)^{-(\alpha_{11}+(2\delta-1)a)/2(\delta-1)a}$.

The similarity variable and the form of similarity solutions for flow variables can be obtained by using invariant surface conditions [2,6] as

$$\rho=t^{(1+\alpha_{11}/a)}\hat{S}(\xi), \tag{3.8}$$

$$u=t^{(\delta-1)}\hat{U}(\xi), \tag{3.9}$$

$$p=t^{(2\delta-1+\alpha_{11}/a)}\hat{P}(\xi), \tag{3.10}$$

$$b=t^{(2\delta-1+\alpha_{11}/a)}\hat{\Pi}(\xi), \tag{3.11}$$

$$q=t^{2(\delta-1)}\hat{E}(\xi)-d. \tag{3.12}$$

The functions $\hat{S}$, $\hat{U}$, $\hat{P}$, $\hat{\Pi}$, and $\hat{E}$ depend only on the similarity variable ξ only, which is given by $\xi=x/At^{\delta}$, where A is a dimensionless constant, whose dimensions are determined by the similarity exponent δ.

Since the shock must be a similar curve and it may be normalized at $\xi=1$. Then, shock path and the shock velocity V are given by $X=At^{\delta}$ and $V=(\delta X/t)$, respectively. And the invariance of the jump condition implies that

$$\begin{gathered} \hat{S}(1)=\frac{\gamma_f+1}{\gamma_f-1}\rho_c\frac{A^{\theta}}{x_0^{\theta}}, \quad \hat{U}(1)=\frac{2\delta A}{\gamma_f+1}, \quad \hat{P}(1)=\frac{2\delta^2\rho_c A^{\theta+2}}{(\gamma_f+1)x_0^{\theta}}, \quad \hat{\Pi}(1)=\frac{1}{2}\left(\frac{\gamma_f+1}{\gamma_f-1}\right)^2\frac{c_0\rho_c\delta^2A^{\theta+2}}{x_0^{\theta}}, \\ \hat{E}(1)=q_0\left(\frac{A}{x_0}\right)^{\Gamma}, \text{ if } q_0 \text{ is varing} \qquad \hat{E}(1)=0, \text{ if } q_0 \text{ is constant}, \\ \Gamma=\frac{2(\delta-1)}{\delta}, \text{ if } q_0 \text{ is varying} \qquad \Gamma=0, \text{ if } q_0 \text{ is constant}, \end{gathered} \tag{3.13}$$

where $\theta=(\alpha_{11}+a/\delta a)$, first relation in (3.13) shows that the necessary condition for the existence of a similarity solution is that the shock cowling number c_0 must be a constant, which implies that θ and δ are not independent but rather $\theta\delta+2(\delta-1)=0$. The form of W given by (2.7) should be consistent with (3.7) in order to have a similarity solution, i.e., if self similar flow patterns possess the similarity conditions

$$ta_1\{\theta\delta+2(\delta-1)\}=-ta_2\theta\delta, \qquad ta_3\{\theta\delta+2(\delta-1)\}=1-ta_4\theta\delta. \tag{3.14}$$

The similarity exponent δ, which is not obtainable from an integral energy balance or the dimensional consideration. It is computed only by solving a nonlinear eigenvalue problem for a system of ordinary differential equations, which is obtained as follows. Substituting (3.13) in the governing equations (2.1)–(2.5) and removing hat notation, we obtained the following system of ordinary differential equations

$$(U-\xi)S' + SU' + \theta S + \frac{SU}{\xi} = 0, \tag{3.15}$$

$$(U-\xi)U' + \frac{\Pi' + P'}{S} + \frac{(\delta-1)}{\delta}U = 0, \tag{3.16}$$

$$(U-\xi)P' + \gamma_f(q)PU' + \theta P + 2\frac{(\delta-1)}{\delta}P + \frac{\gamma_f(q)PU}{\xi} + (\gamma_f(q)-1)SW^* = 0, \tag{3.17}$$

$$(U-\xi)\Pi' + 2\Pi U' + \theta\Pi + 2\frac{(\delta-1)}{\delta}\Pi + \frac{2\Pi U}{\xi} = 0, \tag{3.18}$$

$$(U-\xi)E' + 2\frac{(\delta-1)}{\delta}E - W^* = 0, \tag{3.19}$$

where a prime denotes differentiation with respect to independent variable ξ and

$$W^* = \left\{e_0\delta^{2\beta_1}P^{\beta_1}S^{\beta_2}\left(\frac{P}{S} - \frac{P_0}{\rho_0 V^2}\right) - E + \frac{\hat{q}_c}{\delta^2}\right\}\left(\tau_0\delta^{2\beta_3+1}P^{\beta_3}S^{\beta_2}\right)^{-1}, \quad \text{if } q_0 \text{ is varing,}$$

$$W^* = \left\{e_0\delta^{2\beta_2\beta_1}P^{\beta_1}S^{\beta_2}\left(\frac{P}{S} - \frac{P_0}{\rho_0 V^2}\right) - E\right\}\left(\tau_0\delta^{2\beta_3+1}P^{\beta_3}S^{\beta_4}\right)^{-1}, \quad \text{if } q_0 \text{ is constant,}$$

where $\hat{q}_c$, τ_0, and e_0 are dimensionless parameters defined as

$$\hat{q}_c = q_c A^{\Gamma-2} x_0^{-\Gamma}, \; \tau_0 = \tau^* x^{-\theta(\beta_3+\beta_4)} A^{2\beta_3+\theta(\beta_3+\beta_4)} \rho_c^{(\beta_3+\beta_4)},$$
$$e_0 = c^* x_0^{-\theta(\beta_1+\beta_2)} A^{2\beta_1+\theta(\beta_1+\beta_2)} \rho_c^{\beta_1+\beta_2}, \; c^*, \text{ and } \tau^*$$

are constants with respect to medium.

In view of (3.14), the term $P_0/\rho_0 V^2$ in W^* is either a constant or can be neglected depending on whether the shock is of arbitrary strength or of infinite strength, respectively. Here, we consider the problem of imploding shock for which $V \gg a_0$ in the neighbourhood of implosion and assume q_0 is a constant.

For the problem of a converging shock collapsing to the centre (axis), the origin of time t, is taken to be the instant at which the shock reaches the centre (axis). However, so that $t \leq 0$ in (3.15)–(3.19). In this regard, we modify the definition of the similarity variable by setting $X = A(-t)^\delta$ and $\xi = x/A(-t)^\delta$, so that the interval of the variable are $-\infty < t \leq 0$, $X \leq x < \infty$, and $1 \leq \xi < \infty$.

Thus, the system of equations (3.15)–(3.19) is to be solved subject to the jump conditions at the shock

$$S(1) = \frac{\gamma_f+1}{\gamma_f-1}, \quad U(1) = \frac{2}{\gamma_f+1}, \quad \Pi(1) = c_0\frac{(\gamma_f+1)^2}{2(\gamma_f-1)^2}, \quad P(1) = \frac{2}{\gamma_f+1}, \quad E(1) = 0,$$
$$W^* = e_0\delta^{2\beta_1}P^{1+\beta_1}S^{1-\beta_2} - E^{-e/\tau_0\delta^{2\beta_3+1}P^{\beta_3}S^{\beta_4}}$$

and the boundary conditions at $\xi = \infty$, $U(\infty) = 0$, $P/S|_{\xi=\infty} = 0 = \Pi/S|_{\xi=\infty}$, and $E(\infty) = 0$. In order to find δ and solve the system of equations (3.15)–(3.19), we use the technique given in [8].

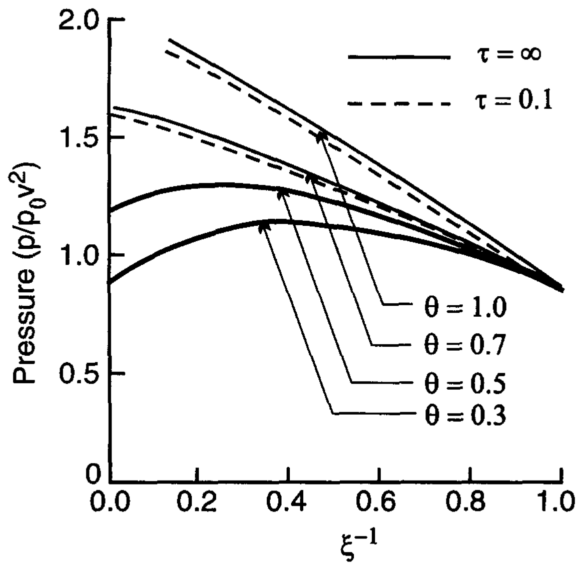

Figure 1. Pressure profile.

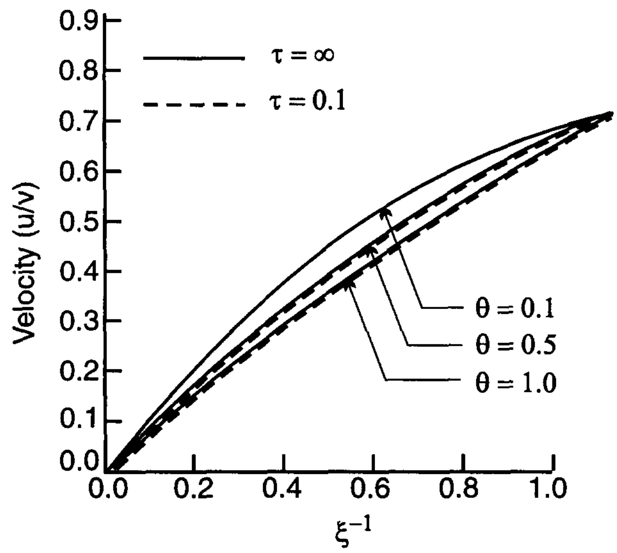

Figure 2. Velocity profile.

4. RESULTS AND DISCUSSION

The presence of magnetic field behind and ahead of the shock has an important effect on the condition at the shock as well as those at the axis of implosion. The results of numerical computations of the flow and field variables at the instant of collapse and before collapse are shown in Figures 1–5.

Depicted results show that the dimensionless gas velocity U decreases monotonically while the dimensionless pressure P, density S, and magnetic pressure Π increase monotonically in the region behind the shock. This is on account of geometric convergence or area contraction of shock wave. The gas pressure and magnetic pressure, which remain bounded in the region behind the shock, gradually increase from the shock to the axis of implosion and attain a maximum there. However, the density profile steepens as the shock converges towards the axis of implosion. The effect of magnetic field on vibrational energy is to slow down the rate of change of vibrational energy in the nonequilibrium flow.

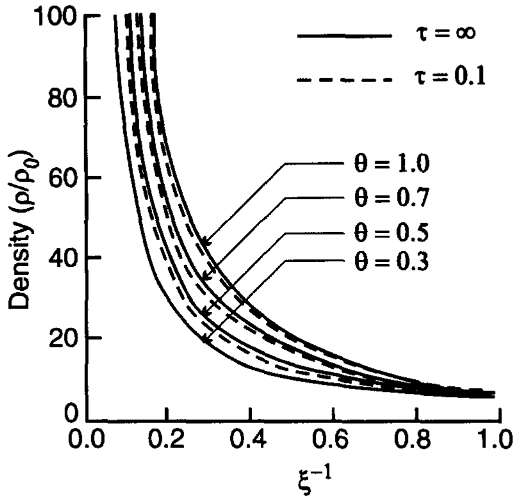

Figure 3. Density profile.

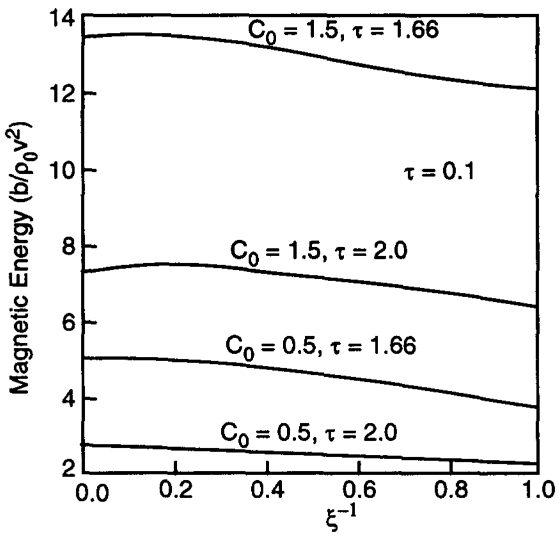

Figure 4. Magnetic pressure profile.

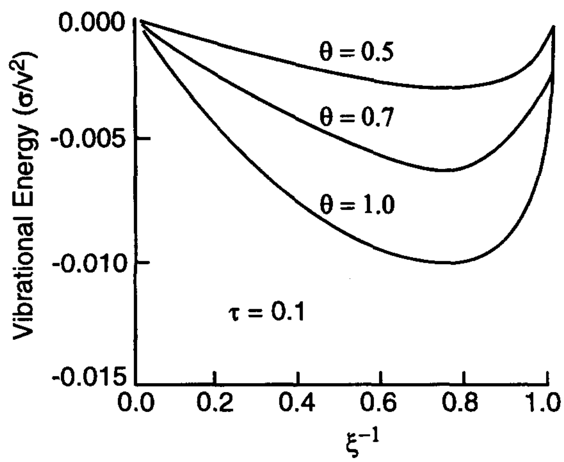

Figure 5. Vibrational energy profile.

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